

Quantifying non-classicality with local unitary operations

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(Dated: February 9, 2012)

We propose a measure of non-classical correlations in bipartite quantum states based on local unitary operations. We prove the measure is non-zero if and only if the quantum discord is non-zero; this is achieved via a new characterization of zero discord states in terms of the state's correlation matrix. Moreover, our scheme can be extended to ensure the same relationship holds even with a generalized version of quantum discord in which higher-rank projective measurements are allowed. We next derive a closed form expression for our scheme in the cases of Werner states and $(2 \times N)$ -dimensional systems. The latter reveals that for two-qubit states, our measure reduces to the geometric discord [Dakić et al., PRL 105, 2010]. A connection to the CHSH inequality is shown. We close with a characterization of all maximally non-classical, yet separable, two-qubit states of rank at most two (with respect to our measure, and hence also with respect to the geometric discord).

PACS numbers: 03.67.Mn, 03.65.Ud

Keywords: Non-classical correlations, quantum discord, local unitary operations

I. INTRODUCTION

One of the most intriguing aspects of quantum mechanics is quantum entanglement, which with the advent of quantum computing, was thrust into the limelight of quantum information theoretic research [1]. We now know that correlations in quantum states due to entanglement are necessary in order for *pure-state* quantum computation to provide exponential speedups over its classical counterpart [2]. With bipartite entanglement nowadays fairly well understood, however, attention has turned in recent years to a more general type of quantum correlation, dubbed simply *non-classical correlations*. Unlike entanglement, such correlations *can* be created via Local Operations and Classical Communication (LOCC), but nevertheless do not exist in the classical setting. Moreover, for certain *mixed-state* quantum computational feats, the amount of entanglement present can be small or vanishing, such as in the DQC1 model of computing [3] and the locking of classical correlations [4]. In these settings, it is rather non-classical correlations which are the conjectured resource enabling such feats (see, e.g. [5–8]). In fact, almost all quantum states possess non-classical correlations [9].

As a result, much attention has recently been devoted to the quantification of non-classical correlations [10–23] (see [24] for a survey). Here, we say a bipartite state ρ acting on Hilbert space $\mathcal{A} \otimes \mathcal{B}$ is *classically correlated* in \mathcal{A} if and only if there exists an orthonormal basis $\{|a_i\rangle\}$ for \mathcal{A} such that

$$\rho = \sum_i p_i |a_i\rangle\langle a_i| \otimes \rho_i$$

for $\{p_i\}$ a probability distribution and ρ_i density operators. To quantify “how far” ρ is from the form above, a number non-classicality measures, including perhaps the best-known such measure, the *quantum discord* [25, 26], ask the question

of how drastically a bipartite quantum state is disturbed under local measurement on \mathcal{A} . In this paper, we take a different approach to the problem. We ask: *Can disturbance of a bipartite system under local unitary operations be used to quantify non-classical correlations?*

It turns out that not only is the answer to this question yes, but that in fact for two-qubit systems, the measure we construct coincides with the *geometric quantum discord* [21], a scheme based again on local measurements. Our measure is defined as follows. Given a bipartite quantum state ρ and unitary U_A acting on Hilbert spaces $\mathcal{A} \otimes \mathcal{B}$ and \mathcal{A} with dimensions MN and M , respectively, define

$$D(\rho, U_A) := \frac{1}{\sqrt{2}} \left\| \rho - (U_A \otimes I_B) \rho (U_A^\dagger \otimes I_B) \right\|_F, \quad (1)$$

where the Frobenius norm $\|A\|_F = \sqrt{\text{Tr} A^\dagger A}$ is used due to its simple calculation. Then, consider the set of unitary operators whose eigenvalues are some permutation of the M -th roots of unity, i.e. whose vector of eigenvalues equals $\pi \mathbf{v}$ for $\pi \in S_M$ some permutation and $v_k = e^{2\pi k i / M}$ for $1 \leq k \leq M$. We call such operators *Root-of-Unity* (RU) unitaries. They include, for example, the Pauli X , Y , and Z matrices. Then, letting $\text{RU}(\mathcal{A})$ denote the set of RU unitaries acting on \mathcal{A} , we define our measure as:

$$D(\rho) := \min_{U_A \in \text{RU}(\mathcal{A})} D(\rho, U_A). \quad (2)$$

Note that $0 \leq D(\rho) \leq 1$ for all ρ acting on $\mathcal{A} \otimes \mathcal{B}$. We now summarize our results regarding $D(\rho)$.

Summary of results and organization of paper

(A) Our first result is a closed-form expression for $D(\rho)$ for $(2 \times N)$ -dimensional systems (Sec. III). This reveals that for two-qubit ρ , $D(\rho)$ coincides with the geometric discord of ρ .

It also allows us to prove that, like the *Fu distance* [27, 28] (defined below in *Previous Work*), if $D(\rho) > 1/\sqrt{2}$, then ρ violates the Clauser-Horne-Shimony-Holt (CHSH) inequality [29].

(B) We next derive a closed form expression for $D(\rho)$ for Werner states (Sec. IV), finding here that $D(\rho)$ in fact equals the Fu distance of ρ .

(C) Sec. V proves that only pure maximally entangled states ρ achieve the maximum value $D(\rho) = 1$, as expected.

(D) In Sec. VI, we show that $D(\rho)$ is a *faithful* non-classicality measure, i.e. it achieves a value of zero if and only if ρ is classically correlated in \mathcal{A} . To prove this, we first derive a new characterization of states with zero quantum discord based on the correlation matrix of ρ . We then show that the states achieving $D(\rho) = 0$ can be characterized in the same way. More generally, by extending our scheme to allow the eigenvalues of U_A to have multiplicity at most k , we prove a state is undisturbed under U_A if and only if there exists a projective measurement on \mathcal{A} of rank at most k acting invariantly on the state (Thm. 10). This reproduces in a simple fashion a result of Ref. [30] regarding entanglement quantification in the pure state setting. Based on this equivalence between disturbance under local unitary operations and local projective measurements, we propose a generalized definition of the quantum discord at the end of Sec. VI.

(E) Finally, we characterize the set of maximally non-classical, yet separable, two-qubit ρ according to $D(\rho)$ (and hence according to the geometric discord) (Sec. VII).

Sec. II begins with necessary definitions and useful lemmas.

Previous work. The Fu distance, defined as the *maximization* of Eqn. (1) over all U_A such that $[U_A, \text{Tr}_B(\rho)] = 0$, was defined in Ref. [27] and studied further in Refs. [28] and [8] with regards to quantifying entanglement and non-classicality. Despite its strengths, such as a closed form solution for two-qubit systems and Werner states, and a connection to the CHSH inequality, the distance has weaknesses: It can attain its maximum value even on non-maximally entangled pure states [28], and is not a faithful non-classicality measure [8]. Interestingly, our $D(\rho)$ eliminates these weaknesses while preserving the former strengths. Subsequent to the conception of our scheme, the present author learned that there has also been an excellent line of work studying (the square of) Eqn. (2) in another setting — that of *pure state entanglement*. In Ref. [31], it was found that in $(2 \times N)$ and $(3 \times N)$ systems, $D(|\psi\rangle\langle\psi|)^2$ coincides with the *linear entropy of entanglement*. Ref. [30] then showed that for arbitrary bipartite pure states, $D(|\psi\rangle\langle\psi|)^2$ is a faithful entanglement monotone, and derived upper and lower bounds in terms of the linear entropy of entanglement. Finally, alternative characterizations of zero discord states have been given in [21, 25, 32]. Maximally non-classical separable two-qubit

states have been studied, for example, in [33, 34].

Open questions and future directions. Our results show that local unitary operations can indeed form the basis of a faithful non-classicality measure with certain desirable properties. We leave open the following questions. For what other interesting classes of quantum states can a closed form expression for $D(\rho)$ be found? Can a better intuitive understanding of the interplay between the notions of “disturbance under local measurements” and “disturbance under local unitary operations” be obtained in higher dimensions? We have given an analytical characterization of all maximally non-classical rank-two separable states — we conjecture that higher rank two-qubit states achieve strictly smaller values of $D(\rho)$. Can this be proven rigorously and analytically? (We remark that a numerical proof for this conjecture was given in [34] for the geometric discord, for example.) What can the study of the generalized notion of quantum discord we defined in Sec. VI, $\delta_v(\rho)$, tell us about non-classical correlations?

II. PRELIMINARIES

We begin by setting our notation, followed by relevant definitions and useful lemmas. Throughout this paper, we use \mathcal{A} and \mathcal{B} to denote complex Euclidean spaces of dimensions M and N , respectively. $\mathcal{D}(\mathcal{A} \otimes \mathcal{B})$, $\mathcal{H}(\mathcal{A} \otimes \mathcal{B})$, and $\mathcal{U}(\mathcal{A} \otimes \mathcal{B})$ denote the sets of density, Hermitian, and unitary operators taking $\mathcal{A} \otimes \mathcal{B}$ to itself, respectively. We define $\rho_A := \text{Tr}_B(\rho)$ and $\rho_B := \text{Tr}_A(\rho)$, where $:=$ indicates a definition. The Frobenius norm of operator A is $\|A\|_F = \text{Tr}(\sqrt{A^\dagger A})$, and the anti-commutator of A and B is $\{A, B\} = AB + BA$. The notation $\text{diag}(\mathbf{v})$ for complex vector \mathbf{v} denotes a diagonal matrix with i th diagonal entry v_i , and $\text{span}(\{\mathbf{v}_i\})$ denotes the span of the set of vectors $\{\mathbf{v}_i\}$. The minimum (maximum) eigenvalue of Hermitian operator A is denoted $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$). Finally, \mathbb{N} is the set of natural numbers.

Moving to definitions, in this paper we often decompose $\rho \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B})$ in terms of a traceless Hermitian basis for $\mathcal{H}(\mathcal{A} \otimes \mathcal{B})$ (sometimes known as the Fano form [35]):

$$\rho = \frac{1}{MN} (I^A \otimes I^B + \mathbf{r}^A \cdot \boldsymbol{\sigma}^A \otimes I^B + I^A \otimes \mathbf{r}^B \cdot \boldsymbol{\sigma}^B + \sum_{i=1}^{M^2-1} \sum_{j=1}^{N^2-1} T_{ij} \sigma_i^A \otimes \sigma_j^B). \quad (3)$$

Here, $\boldsymbol{\sigma}^A$ is a $(M^2 - 1)$ -component vector of traceless orthogonal Hermitian basis elements σ_i^A satisfying $\text{Tr}(\sigma_i^A \sigma_j^A) = 2\delta_{ij}$, $\mathbf{r}^A \in \mathbb{R}^{M^2-1}$ is the Bloch vector for subsystem A with $r_i^A = \frac{M}{2} \text{Tr}(\rho_A \sigma_i^A)$, and $T \in \mathbb{R}^{(M^2-1) \times (N^2-1)}$ is the correlation matrix with entries $T_{ij} = \frac{MN}{4} \text{Tr}(\sigma_i^A \otimes \sigma_j^B \rho)$. For $M = 2$, \mathbf{r}_A satisfies $0 \leq \|\mathbf{r}_A\|_2 \leq 1$ with $\|\mathbf{r}_A\|_2 = 1$ if and only if ρ_A is pure. The definitions for subsystem B are analogous. We now give a useful specific construction for the basis

elements σ_i^A [36]. Define $\{\sigma_i\}_{i=1}^{M^2-1} = \{U_{pq}, V_{pq}, W_r\}$, such that for $1 \leq p < q \leq M$ and $1 \leq r \leq M-1$, and $\{|i\rangle\}_{i=1}^M$ some orthonormal basis for \mathcal{A} :

$$U_{pq} = |p\rangle\langle q| + |q\rangle\langle p| \quad (4)$$

$$V_{pq} = -i|p\rangle\langle q| + i|q\rangle\langle p| \quad (5)$$

$$W_r = \sqrt{\frac{2}{r(r+1)}} \left(\sum_{k=1}^r |k\rangle\langle k| - r|r+1\rangle\langle r+1| \right). \quad (6)$$

Note that when $M = 2$, this construction yields the Pauli matrices $\sigma^A = (X, Y, Z)$.

Regarding $D(\rho)$, defining $\rho_f := (U_A \otimes I_B)\rho(U_A^\dagger \otimes I_B)$, we often use the fact that Eqn. (2) can be rewritten as:

$$D(\rho) = \min_{U_A \in \text{RU}(\mathcal{A})} \sqrt{\text{Tr}(\rho^2) - \text{Tr}(\rho\rho_f)}. \quad (7)$$

Finally, we show a simple but important lemma.

Lemma 1. $D(\rho)$ is invariant under local unitary operations.

Proof. Let $\rho' := (V_A \otimes V_B)\rho(V_A \otimes V_B)^\dagger$ for unitaries V_A, V_B . Then in Eqn. (7), $\text{Tr}(\rho'^2) = \text{Tr}(\rho^2)$, and $\text{Tr}(\rho'\rho'_f)$ becomes

$$\text{Tr}(\rho(V_A^\dagger U_A V_A \otimes I_B)\rho(V_A^\dagger U_A^\dagger V_A \otimes I_B)).$$

Observe, however, that $V_A U_A V_A^\dagger$ is still an RU unitary, since we have simply changed basis. Hence, $D(\rho', U_A) = D(\rho, V_A^\dagger U_A V_A)$, and since we are minimizing over all $U_A \in \text{RU}(\mathcal{A})$, the claim follows. \square

III. $(2 \times N)$ -DIMENSIONAL STATES

In this section, we study $D(\rho)$ for $\rho \in \mathcal{D}(\mathbb{C}^2 \otimes \mathbb{C}^N)$, obtaining among other results a closed form expression for $D(\rho)$. To begin, note that any $U_A \in \text{RU}(\mathcal{A})$ must have the form

$$U_A := |c\rangle\langle c| - |d\rangle\langle d| = 2|c\rangle\langle c| - I_2, \quad (8)$$

up to an irrelevant global phase which disappears upon application of U_A to our system, and for some orthonormal basis $\{|c\rangle, |d\rangle\}$ for \mathbb{C}^2 . Then, $D(\rho, U_A)$ can be rewritten as

$$2\sqrt{\text{Tr}[\rho^2(|c\rangle\langle c| \otimes I) - \rho(|c\rangle\langle c| \otimes I)\rho(|c\rangle\langle c| \otimes I)]}. \quad (9)$$

We begin with a simple upper bound on $D(\rho)$.

Theorem 1. For any $\rho \in \mathcal{D}(\mathbb{C}^2 \otimes \mathbb{C}^N)$, one has

$$D(\rho) \leq 2\sqrt{\lambda_{\min}(\text{Tr}_B(\rho^2))}.$$

Proof. Starting with Eqn. (9), by noting that $\text{Tr}[\rho(|c\rangle\langle c| \otimes I)\rho(|c\rangle\langle c| \otimes I)] \geq 0$ and using the fact that $\text{Tr}(\rho(C_A \otimes I_B)) = \text{Tr}(\rho_A C_A)$, we have that $D(\rho)$ is at most

$$\min_{\text{unit } |c\rangle \in \mathbb{C}^2} 2\sqrt{\text{Tr}[\text{Tr}_B(\rho^2)|c\rangle\langle c|]} = 2\sqrt{\lambda_{\min}(\text{Tr}_B(\rho^2))}. \quad \square$$

Thm. 1 implies that for pure product $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^N$, $D(|\psi\rangle\langle\psi|) = 0$, in agreement with the results in Ref. [31]. By next exploiting the structure of ρ further, we obtain a closed form expression for $D(\rho)$.

Theorem 2. For any $\rho \in \mathcal{D}(\mathbb{C}^2 \otimes \mathbb{C}^N)$, $D(\rho)$ equals

$$\frac{1}{\sqrt{N}} \sqrt{\|\mathbf{r}^A\|_2^2 + \frac{2}{N} \sum_{i,j=1}^3 T_{ij}^2 - \lambda_{\max} \left(\mathbf{r}^A (\mathbf{r}^A)^T + \frac{2}{N} T T^T \right)}. \quad (10)$$

Proof. Define $P := |c\rangle\langle c|$. Then, beginning with Eqn. (9), by rewriting ρ using Eqn. (3) and applying the fact that the basis elements σ_i are traceless, we obtain that $\text{Tr}(\rho^2 P \otimes I - \rho P \otimes I \rho P \otimes I)$ equals

$$\frac{1}{4N} \text{Tr}(A_1 - A_2 + A_3 - A_4),$$

where

$$A_1 := \left(\sum_i r_i^A \sigma_i^A \right)^2 P$$

$$A_2 := \left(\sum_i r_i^A \sigma_i^A P \right)^2$$

$$A_3 := \frac{1}{N} \left(\sum_{ij} T_{ij} \sigma_i^A \otimes \sigma_j^B \right)^2 (P \otimes I)$$

$$A_4 := \frac{1}{N} \left(\sum_{ij} T_{ij} \sigma_i^A \otimes \sigma_j^B \right) \left(\sum_{ij} T_{ij} P \sigma_i^A P \otimes \sigma_j^B \right).$$

Using the facts that $(\sigma_i^A)^2 = I$, $\{\sigma_i^A, \sigma_j^A\} = 0$ for $i \neq j$, $\text{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$, and $\text{Tr}(P) = 1$, we thus have

$$\text{Tr}(A_1) = \|\mathbf{r}^A\|_2^2, \quad \text{Tr}(A_3) = \frac{2}{N} \sum_{ij} T_{ij}^2$$

$$\text{Tr}(A_2) = \sum_{ij} r_i^A r_j^A \langle c | \sigma_i^A | c \rangle \langle c | \sigma_j^A | c \rangle$$

$$\text{Tr}(A_4) = \frac{2}{N} \sum_{ij} \left(\sum_k T_{ik} T_{jk} \right) \langle c | \sigma_i^A | c \rangle \langle c | \sigma_j^A | c \rangle.$$

Now, $\langle c | \sigma_i^A | c \rangle$ can be thought of as the i th component of the Bloch vector $\mathbf{v} \in \mathbb{R}^3$ of pure state $|c\rangle$ with $\|\mathbf{v}\|_2 = 1$, implying

$$\text{Tr}(A_2 + A_4) = \mathbf{v}^T \left[\mathbf{r}^A (\mathbf{r}^A)^T + \frac{2}{N} T T^T \right] \mathbf{v}.$$

Plugging these values into Eqn. (9), we conclude $D(\rho)$ equals

$$\min_{\substack{\mathbf{v} \in \mathbb{R}^3 \\ \|\mathbf{v}\|_2=1}} \frac{1}{\sqrt{N}} \sqrt{\|\mathbf{r}^A\|_2^2 + \frac{2}{N} \sum_{ij} T_{ij}^2 - \text{Tr}(A_2 + A_4)}.$$

The claim now follows since for any symmetric $A \in \mathbb{R}^{n \times n}$, $\max_{\text{unit } \mathbf{v} \in \mathbb{R}^n} \mathbf{v}^T A \mathbf{v} = \lambda_{\max}(A)$. \square

Note that when $N = 2$, the expression for $D(\rho)$ in Thm. 2 matches that for the *geometric discord* [21]. Specifically, defining the latter as $\delta_g(\rho) = \min_{\sigma \in \Omega} \sqrt{2} \|\rho - \sigma\|_F$, where Ω is the set of zero-discord states, we have for two-qubit ρ that $D(\rho) = \delta_g(\rho)$. (Note: The original definition of Ref. [21] was more precisely $\delta_g(\rho) = \min_{\sigma \in \Omega} \|\rho - \sigma\|_F^2$.) We now discuss consequences of Thm. 2, beginning with a lower bound which proves useful later.

Corollary 3. *For $\rho \in \mathcal{D}(\mathbb{C}^2 \otimes \mathbb{C}^N)$, we have*

$$D(\rho) \geq \frac{\sqrt{2}}{N} \sqrt{\sum_{i,j=1}^3 T_{ij}^2 - \lambda_{\max}(TT^T)}. \quad (11)$$

This holds with equality if $\mathbf{r}^A = 0$, i.e. $\rho_A = \frac{I}{2}$.

Proof. The first claim follows from the fact that:

$$\lambda_{\max}\left(\mathbf{r}^A(\mathbf{r}^A)^T + \frac{2}{N}TT^T\right) \leq \|\mathbf{r}^A\|_2^2 + \frac{2}{N}\lambda_{\max}(TT^T).$$

The second claim follows by substitution into Eqn. (10). \square

For example, for maximally entangled $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, for which $\mathbf{r}^B = \mathbf{0}$ and $T = \text{diag}(1, -1, 1)$, Cor. 3 yields $D(|\psi\rangle\langle\psi|) = 1$, as desired. We also remark that Eqn. (10) can further be simplified for two-qubit states, since by Ref. [37, 38], one can assume without loss of generality that T is diagonal. This relies on the facts that (1) applying local unitary $V_1 \otimes V_2$ to ρ has the effect of mapping $T \mapsto O_1 T O_2^\dagger$, $\mathbf{r}^A \mapsto O_1 \mathbf{r}^A$, and $\mathbf{r}^B \mapsto O_2 \mathbf{r}^B$ for some orthogonal rotation matrices O_1 and O_2 , and (2) $D(\rho)$ is invariant under local unitaries by Lem. 1.

Using Cor. 3, we next obtain a connection to the CHSH inequality for two-qubit ρ . Defining $M(\rho) := \lambda_1(T^T T) + \lambda_2(T^T T)$, where $\lambda_i(C)$ denotes the i th largest eigenvalue of Hermitian matrix C , it is known that ρ violates the CHSH inequality if and only if $M(\rho) > 1$ [39]. We thus have:

Corollary 4. *For $\rho \in \mathcal{D}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, if $D(\rho) > 1/\sqrt{2}$, then $M(\rho) > 1$. The converse does not hold.*

Proof. The claim follows from Cor. 3 by observing that

$$\sum_{ij} T_{ij}^2 = \text{Tr}(TT^T) = \sum_{i=1}^3 \lambda_i(TT^T),$$

and then applying the fact that TT^T and $T^T T$ are cospectral (Thm. 1.3.20 of [40]). The converse proceeds similarly to Thm. 7 of Ref. [28] — namely, let $|\psi\rangle = a|00\rangle + b|11\rangle$ for real $a, b \geq 0$ and $a^2 + b^2 = 1$. Then, for density operator $|\psi\rangle\langle\psi|$, we have $\mathbf{r}^B = (0, 0, a^2 - b^2)$ and $T = \text{diag}(2ab, -2ab, 1)$, implying $M(|\psi\rangle\langle\psi|) > 1$ for $a, b \neq 0$. In comparison, $D(|\psi\rangle\langle\psi|) = 2ab \leq 1/\sqrt{2}$ when $a \leq \sqrt{\frac{1}{2} - \frac{1}{2\sqrt{2}}}$ or $a \geq \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}}$. \square

Interestingly, the exact same relationship as that in Cor. 4 was found between the Fu distance and the CHSH inequality in Ref. [28].

IV. WERNER STATES

We now derive a closed formula for $D(\rho)$ for Werner states $\rho \in \mathcal{D}(\mathbb{C}^d \otimes \mathbb{C}^d)$ where $d \geq 2$, which are defined as [41]

$$\rho := \frac{2p}{d^2 + d} P_s + \frac{2(1-p)}{d^2 - d} P_a,$$

for $P_s := (I + P)/2$ and $P_a := (I - P)/2$ the projectors onto the symmetric and anti-symmetric subspaces, respectively, $P := \sum_{i,j=1}^d |i\rangle\langle j| \otimes |j\rangle\langle i|$ the SWAP operator, and $0 \leq p \leq 1$. Werner states are invariant under $U \otimes U$ for any unitary U , and are entangled if and only if $p < 1/2$.

Theorem 5. *Let $\rho \in \mathcal{D}(\mathbb{C}^d \otimes \mathbb{C}^d)$ be a Werner state. Then*

$$D(\rho) = \frac{|2pd - d - 1|}{d^2 - 1}.$$

Proof. As done in Thm. 3 of Ref. [28], we first rewrite Eqn. 7 using the facts that $\text{Tr}(P) = d$, $\text{Tr}(P^2) = d^2$, and $\beta := \text{Tr}(P(U_A \otimes I)P(U_A \otimes I)^\dagger) = \text{Tr}(U_A)\text{Tr}(U_A^\dagger)$ to obtain that for any $U_A \in U(\mathcal{A})$,

$$D(\rho, U_A) = \frac{\sqrt{(2pd - d - 1)^2(d^2 - \beta)}}{d(d^2 - 1)}.$$

Since $\text{Tr}(U_A) = 0$ for any $U_A \in \text{RU}(\mathcal{A})$, we have $\beta = 0$ and the claim follows. \square

Again, we find that this coincides exactly with the expression for the Fu distance for Werner states [28]. Further, Thm. 5 implies that the quantum discord of Werner state ρ is zero if and only if $p = (d+1)/2d$. This matches the results of Chitambar [42], who develops the following closed formula for the discord $\delta(\rho)$ of Werner states:

$$\delta(\rho) = \log(d+1) + (1-p) \log \frac{1-p}{d-1} + p \log \frac{p}{d+1} - \frac{2p}{d+1} \log p - \left(1 - \frac{2p}{d+1}\right) \log \frac{d+1-2p}{2(d-1)}. \quad (12)$$

In Sec. VI, we show that this is no coincidence — it turns out that $D(\rho) = 0$ if and only if the discord of ρ is zero for any ρ .

V. PURE STATES OF ARBITRARY DIMENSION

We now show that only pure maximally entangled states ρ achieve $D(\rho) = 1$. As mentioned in Sec. I, this is in contrast to the Fu distance [27, 28], whose maximal value is attained even for certain *non-maximally* entangled $|\psi\rangle$. We remark that

Thm. 6 below also follows from a more general non-trivial result that $D(|\psi\rangle\langle\psi|)^2$ is tightly upper bounded by the linear entropy of entanglement of pure state $|\psi\rangle$ [30]. However, our proof of Thm. 6 is much simpler and requires only elementary linear algebra.

To begin, assume without loss of generality that $M \leq N$, and let $|\psi\rangle \in \mathcal{A} \otimes \mathcal{B}$ be a pure quantum state with Schmidt decomposition $|\psi\rangle = \sum_{k=1}^M \alpha_k |a_k\rangle \otimes |b_k\rangle$, i.e. $\sum_k \alpha_k^2 = 1$ for $\alpha_k \in \mathbb{R}$ and $\{|a_k\rangle\}$ and $\{|b_k\rangle\}$ the Schmidt bases for \mathcal{A} and \mathcal{B} , respectively.

Theorem 6. *Let $|\psi\rangle \in \mathcal{A} \otimes \mathcal{B}$ with Schmidt decomposition as above. Then $D(|\psi\rangle\langle\psi|) = 1$ if and only if $\alpha_k = \frac{1}{\sqrt{M}}$ for all $1 \leq k \leq M$ (i.e. $|\psi\rangle$ is maximally entangled).*

Proof. We begin by rewriting Eqn. (7) as

$$D(|\psi\rangle\langle\psi|) = \min_{U_A \in \text{RU}(\mathcal{A})} \sqrt{1 - \left| \sum_{k=1}^M \alpha_k^2 \langle a_k | U_A | a_k \rangle \right|^2}. \quad (13)$$

If $|\psi\rangle$ is maximally entangled, then $\alpha_k = 1/\sqrt{M}$ for all $1 \leq k \leq M$. Then, since $U_A \in \text{RU}(\mathcal{A})$, Eqn. (13) yields

$$D(|\psi\rangle\langle\psi|) = \min_{U_A \in \text{RU}(\mathcal{A})} \sqrt{1 - \frac{1}{M^2} |\text{Tr}(U_A)|^2} = 1.$$

For the converse, assume $D(|\psi\rangle\langle\psi|) = 1$. Then, by Eqn. (13), we must have that for all $U_A \in \text{RU}(\mathcal{A})$,

$$\sum_{k=1}^M \alpha_k^2 \langle a_k | U_A | a_k \rangle = 0. \quad (14)$$

Thus, choosing U_A as diagonal in basis $\{|a_k\rangle\}$, Eqn. (14) equivalently says that $\mathbf{w}^T \pi \mathbf{v} = 0$ for all permutations $\pi \in S_M$, where $w_k := \alpha_k^2$ and $v_k := e^{2\pi k i/M}$. This can only hold, however, if all entries of \mathbf{w} are the same, i.e. $\alpha_k = 1/\sqrt{M}$ for all $1 \leq k \leq M$, as desired. \square

Corollary 7. *A quantum state $\rho \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B})$ achieves $D(\rho) = 1$ if and only if ρ is pure and maximally entangled.*

Proof. Immediate from Thm. 6 and the $\text{Tr}(\rho^2)$ in Eqn. (7). \square

VI. RELATIONSHIP TO QUANTUM DISCORD

We now show that for arbitrary $\rho \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B})$, $D(\rho)$ is zero if and only if the quantum discord of ρ is zero. The discord is defined as follows [25]:

$$\delta(\rho) := S(\mathcal{A}) - S(\mathcal{A}, \mathcal{B}) + \min_{\{\Pi_j^A\}} S(\mathcal{B} | \{\Pi_j^A\}), \quad (15)$$

where $\{\Pi_j^A\}$ corresponds to a complete measurement on subsystem \mathcal{B} consisting of rank 1 projectors, $S(\mathcal{B}) =$

$-\text{Tr}(\rho_B \log(\rho_B))$ is the von Neumann entropy of ρ_B , similarly $S(\mathcal{A}, \mathcal{B}) = S(\rho)$, and

$$S(\mathcal{B} | \{\Pi_j^A\}) = \sum_j p_j S\left(\frac{1}{p_j} \Pi_j^A \otimes I^B \rho \Pi_j^A \otimes I^B\right), \quad (16)$$

where $p_j = \text{Tr}(\Pi_j^A \otimes I^B \rho)$. Here, the main fact we leverage about the discord is the following.

Theorem 8 (Ollivier and Zurek [25]). *For $\rho \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B})$, $\delta(\rho) = 0$ if and only if*

$$\rho = \sum_j \Pi_j^A \otimes I^B \rho \Pi_j^A \otimes I^B, \quad (17)$$

for some complete set of rank 1 projectors $\{\Pi_j^A\}$.

We now prove the main result of this section. The first part of the proof involves a new characterization of the set of zero discord quantum states ρ in terms of the basis elements σ_i^A from the Fano form of ρ . Key to this characterization is the absence of non-diagonal σ_i^A in the expansion of ρ . In the proofs below, we assume the basis elements σ_i^A for \mathcal{A} come from the set $\{I, U_{pq}, V_{pq}, W_r\}_{p,q,r}^A$ from Sec. II (analogously for \mathcal{B}).

Theorem 9. *Let $\rho \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B})$. Then $\delta(\rho) = 0$ if and only if there exists a local unitary V^A such that*

$$\text{Tr}\left((V^A \otimes I^B) \rho (V^{A\dagger} \otimes I^B) (\sigma_i^A \otimes \sigma_j^B)\right) = 0$$

for all $\sigma_i^A \in \{U_{pq}, V_{pq}\}^A$ and all $\sigma_j^B \in \{I, U_{pq}, V_{pq}, W_r\}^B$. The same characterization holds for $D(\rho) = 0$.

Proof. We prove the equivalent statement that $\delta(\rho) = 0$ if and only if there exists an orthonormal basis $\{|k\rangle\}$ for \mathcal{A} such that, for basis elements σ_i^A constructed with respect to $\{|k\rangle\}$, we have $\text{Tr}(\rho(\sigma_i^A \otimes \sigma_j^B)) = 0$ for all $\sigma_i^A \in \{U_{pq}, V_{pq}\}$ (and similarly for $D(\rho) = 0$).

Suppose $\delta(\rho) = 0$. Then by Thm. 8, there exists a complete set of rank 1 projectors $\{\Pi_j^A\}$ such that Eqn. (17) holds. Let $\{|k\rangle\}$ be the basis onto which $\{\Pi_j^A\}$ projects, and define $\Phi(C) := \sum_j \Pi_j^A C \Pi_j^A$. By constructing the basis elements σ_i^A in Eqn. (3) using $\{|k\rangle\}$, we thus have

$$\rho = \frac{1}{MN} [I^A \otimes I^B + I^A \otimes \mathbf{r}^B \cdot \sigma^B + \sum_{i=1}^{M^2-1} \Phi(\sigma_i^A) \otimes \left(r_i^A I^B + \sum_{j=1}^{N^2-1} T_{ij} \sigma_j^B \right)]. \quad (18)$$

Now, for all $\sigma_i^A \in \{W_r\}$, we clearly have $\Phi(\sigma_i^A) = \sigma_i^A$. For $\sigma_i^A \in \{U_{pq}, V_{pq}\}$, however, $\Phi(\sigma_i^A) = 0$. Thus, in order for Eqn. (17) to hold, we must have $r_i^A = T_{ij} = 0$ for all basis elements $\sigma_i^A \in \{U_{pq}, V_{pq}\}$, which by definition means $\text{Tr}(\rho(\sigma_i^A \otimes \sigma_j^B)) = 0$ for all $\sigma_i^A \in \{U_{pq}, V_{pq}\}^A$, as desired. To show that this implies $D(\rho) = 0$, construct $U^A \in \text{RU}(\mathcal{A})$

as diagonal in basis $\{|k\rangle\}$ and define $\Phi(C) := U^A C U^{A\dagger}$. Then since in Eqn. (18), we have $\Phi(\sigma_i^A) = \sigma_i^A$ for any $\sigma_i^A \in \{I, W_r\}$, the claim follows.

To show the converse, assume $D(\rho, U^A) = 0$ for some $U^A \in \text{RU}(\mathcal{A})$. Then, construct the basis elements σ_i^A with respect to a diagonalizing basis $\{|k\rangle\}$ for U^A and define $\Phi(C) := U^A C U^{A\dagger}$. It follows that for any p and q ,

$$\Phi(U_{pq}) = e^{i(\theta_p - \theta_q)} |p\rangle\langle q| + e^{-i(\theta_p - \theta_q)} |q\rangle\langle p|, \quad (19)$$

$$\Phi(V_{pq}) = -ie^{i(\theta_p - \theta_q)} |p\rangle\langle q| + ie^{-i(\theta_p - \theta_q)} |q\rangle\langle p|. \quad (20)$$

Consider now an arbitrary term $(c_u \sigma_u^A + c_v \sigma_v^A) \otimes \sigma_j^B$ from the Fano form of ρ where $\sigma_u^A = U_{pq}$ and $\sigma_v^A = V_{pq}$ for some choice of p and q . Since Eqns. (19) and (20) imply that U^A can only map U_{pq} to V_{pq} and vice versa, it follows that in order for $D(\rho, U^A) = 0$ to hold, we must have $\Phi(c_u \sigma_u^A + c_v \sigma_v^A) = c_u \sigma_u^A + c_v \sigma_v^A$. This leads to the system of equations

$$\begin{aligned} c_u - ic_v &= e^{i(\theta_p - \theta_q)} (c_u - ic_v) \\ c_u + ic_v &= e^{-i(\theta_p - \theta_q)} (c_u + ic_v). \end{aligned}$$

We conclude that if either $c_u \neq 0$ or $c_v \neq 0$, it must be that $\theta_p = \theta_q$ in order for $D(\rho) = 0$ to hold. However, since all eigenvalues of U^A are distinct by definition, this is impossible. Thus, $\text{Tr}(\rho(\sigma_i^A \otimes \sigma_j^B)) = 0$ for all $\sigma_i^A \in \{U_{pq}, V_{pq}\}$, as desired. To see that this implies $\delta(\rho) = 0$, simply now choose $\{\Pi_j^A\}$ as the projection onto $\{|k\rangle\}$. Then, defining $\Phi(C) := \sum_j \Pi_j^A C \Pi_j^A$ and applying the same arguments from the forward direction to Eqn. (18), we conclude that ρ is invariant under $\{\Pi_j^A\}$. By Thm. 8, we have $\delta(\rho) = 0$, completing the proof. \square

Thus, unlike the Fu distance [8], $D(\rho)$ is indeed a *faithful* non-classicality measure. The proof of Thm. 9 does, however, have a curiosity — in order to achieve this faithfulness, it would have sufficed to minimize over $U^A \in \mathcal{U}(\mathcal{A})$ with merely *distinct* eigenvalues, as opposed to $U^A \in \text{RU}(\mathcal{A})$. Interestingly, this is the mixed-state analogue of the pure-state result of Ref. [30], where it was shown that distinct eigenvalues suffice to conclude $D(|\psi\rangle\langle\psi|)$ is a faithful entanglement monotone for pure states $|\psi\rangle$. More generally, Ref. [30] shows that if U^A has eigenvalues of multiplicity at most k (with at least one eigenvalue of multiplicity k), then $D(|\psi\rangle\langle\psi|) = 0$ if and only if $|\psi\rangle$ has Schmidt rank at most k . Could there be an analogue of this more general result in the mixed-state setting of non-classicality? It turns out the answer is yes.

Let $\mathbf{v} \in \mathbb{N}^M$ such that $\sum_{j=1}^M v_j j = M$. Then, consider an arbitrary (i.e. not necessarily RU) unitary $U_{\mathbf{v}}^A$ which has precisely v_j distinct eigenvalues with multiplicity j . For example, $U_{\mathbf{v}}^A \in \text{RU}(\mathcal{A})$ has $\mathbf{v} = (M, 0, \dots, 0)$ since it has M distinct eigenvalues of multiplicity 1. Similarly, if $\mathbf{v} = (0, 0, \dots, 1)$, then $U_{\mathbf{v}}^A$ is just the identity (up to phase), and if $\mathbf{v} = (M-4, 2, \dots, 0)$ then $U_{\mathbf{v}}^A$ has $M-4$ distinct eigenvalues

of multiplicity 1, and two distinct eigenvalues with multiplicity 2 each. Now, corresponding to any $U_{\mathbf{v}}^A$ is a complete projective measurement $\{\Pi_j^A\}_{\mathbf{v}}$ which consists precisely of v_j projectors of rank j . The correspondence is simple: Let λ be an eigenvalue of $U_{\mathbf{v}}^A$ with multiplicity j , i.e. the projector Π_{λ} onto its eigenspace has rank j . Then $\Pi_{\lambda} \in \{\Pi_j^A\}_{\mathbf{v}}$. It is easy to see that similarly, corresponding to any $\{\Pi_j^A\}_{\mathbf{v}}$ is a $U_{\mathbf{v}}^A$ (assuming we are not concerned with the precise eigenvalues of $U_{\mathbf{v}}^A$, as is this case here). We can now state the following.

Theorem 10. *Let $\rho \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B})$ and $\mathbf{v} \in \mathbb{N}^M$ such that $\sum_{j=1}^M v_j j = M$. Then, there exists a complete projective measurement $\{\Pi_j^A\}_{\mathbf{v}}$ such that*

$$\rho = \sum_j \Pi_j^A \otimes I^B \rho \Pi_j^A \otimes I^B \quad (21)$$

if and only if there exists a $U_{\mathbf{v}}^A \in \mathcal{U}(\mathcal{A})$ with $D(\rho, U_{\mathbf{v}}^A) = 0$.

Proof. The proof follows that of Thm. 9, so we outline the differences. Here, $U_{\mathbf{v}}^A$ and $\{\Pi_j^A\}_{\mathbf{v}}$ will be related through the correspondence outlined above, and the basis elements σ_i^A are constructed with respect to a diagonalizing basis $\{|k\rangle\}$ for $U_{\mathbf{v}}^A$ (which by definition also diagonalizes each $\Pi_j^A \in \{\Pi_j^A\}_{\mathbf{v}}$). For simplicity, we discuss the case of $\mathbf{v} = (M-2, 1, 0, \dots, 0)$; all other cases proceed analogously.

Going in the forward direction, suppose $\Pi_j^A \in \{\Pi_j^A\}_{\mathbf{v}}$ projects onto $\mathcal{S}_{pq} := \text{span}(|p\rangle, |q\rangle)$. Then, in Eqn. (18), $\Phi(\sigma_i^A) = \sigma_i^A$ for $\sigma_i^A = U_{pq}$ and $\sigma_i^A = V_{pq}$. In other words, now we can have $r_i^A \neq 0$ and $T_{ij} \neq 0$ (however, note we still have $r_{m \neq i}^A = 0$ and $T_{m \neq i, j} = 0$). Since $U_{\mathbf{v}}^A$ has a degenerate eigenvalue on \mathcal{S}_{pq} , however, we have by Eqns. (19) and (20) that $U_{\mathbf{v}}^A$ acts invariantly on σ_i^A as well (since $\theta_p = \theta_q$). The converse is similar; namely, suppose $U_{\mathbf{v}}^A$ has a degenerate eigenvalue on \mathcal{S}_{pq} . Then the projector onto the corresponding two-dimensional eigenspace $\Pi_j^A \in \{\Pi_j^A\}_{\mathbf{v}}$ is $\Pi_j^A = |p\rangle\langle p| + |q\rangle\langle q|$. It thus follows by the same argument as above that both $U_{\mathbf{v}}^A$ and Π_j^A act invariantly on U_{pq} and V_{pq} . \square

From this general theorem, we can re-derive as a simple corollary the pure state result of Ref. [30] mentioned earlier, which we rephrase in our terminology as follows.

Corollary 11. *Let $|\psi\rangle = \sum_{i=1}^r \alpha_i |\psi_i^A\rangle |\psi_i^B\rangle$ be the Schmidt decomposition of $|\psi\rangle \in \mathcal{A} \otimes \mathcal{B}$. Then, there exists $U_{\mathbf{v}}^A \in \mathcal{U}(\mathcal{A})$ with $v_k \geq 1$ (i.e. $U_{\mathbf{v}}^A$ has an eigenvalue of multiplicity k), $v_{k' > k} = 0$ (all eigenvalues of $U_{\mathbf{v}}^A$ have multiplicity at most k), and $D(|\psi\rangle\langle\psi|, U_A) = 0$ if and only if $k \geq r$.*

Proof. Suppose $k \geq r$. Then, by defining $\{\Pi_j^A\}_{\mathbf{v}}^k$ such that $v_k \geq 1$ and $v_{k' > k} = 0$, one can choose a $\{\Pi_j^A\}_{\mathbf{v}}^k$ such that Eqn. (21) holds for $\rho = |\psi\rangle\langle\psi|$ (i.e. simply project onto $\text{span}(\{|\psi_i^A\rangle\})$). By Thm. 10, this implies there exists a $U_{\mathbf{v}}^A$ with $v_k \geq 1$ and $v_{k' > k} = 0$ achieving $D(|\psi\rangle\langle\psi|, U_A) = 0$.

Conversely, if $k < r$, then clearly no such $\{\Pi_j^A\}_{\mathbf{v}}^k$ such that Eqn. (21) holds exists. By Thm. 10, this implies that no U_A with an eigenvalue of multiplicity at most k and $D(|\psi\rangle\langle\psi|, U_A) = 0$ exists, as desired. \square

We close this section with two final comments. First, given Thm. 9, one might ask whether a stronger relationship between $D(\rho)$ and $\delta(\rho)$ holds. For example, could it be that $D(\rho) \geq \delta(\rho)$ for all ρ ? This simplest type of relationship is ruled out easily via Thm. 5 and Eqn. (12), since for $d = 2$ and $p = 2/3$, $D(\rho) = 1/9 \geq \delta(\rho) \approx 0.01614$, while for $d = 50$ and $p = 2/3$, $D(\rho) \approx 0.00627 \leq \delta(\rho) \approx 0.07111$.

Second, note that Thm. 10 reduces to Thm. 9 if we choose $\mathbf{v} = (M, 0, \dots, 0)$. This suggests defining a *generalized quantum discord*, denoted $\delta_{\mathbf{v}}(\rho)$, which is analogous to $\delta(\rho)$, except that now we use the class of measurements $\{\Pi_j^A\}_{\mathbf{v}}$ in Eqn. (15). For example, $\delta_{(M, 0, \dots, 0)}(\rho) = \delta(\rho)$. We hope the study of $\delta_{\mathbf{v}}(\rho)$ would prove fruitful in its own right.

VII. MAXIMALLY NON-CLASSICAL SEPARABLE TWO QUBIT STATES

In this section, we characterize the set of maximally non-classical, yet separable, two qubit states of rank at most two, as quantified by $D(\rho)$. To do so, consider separable state

$$\rho = \sum_{i=1}^n p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \quad (22)$$

where $\sum_i p_i = 1$, $|a_i\rangle \in \mathbb{C}^2$, $|b_i\rangle \in \mathbb{C}^2$. Via simple algebraic manipulation, one then finds that $D(\rho, U_A)$ for any given $U_A \in \text{RU}(\mathcal{A})$ is given by

$$\sqrt{\sum_{i=1}^n \sum_{j=1}^n p_i p_j |\langle b_i | b_j \rangle|^2 (|\langle a_i | a_j \rangle|^2 - |\langle a_i | U_A | a_j \rangle|^2)}. \quad (23)$$

We begin by proving a simple but useful upper bound on $D(\rho)$ which depends solely on n .

Lemma 2. *Let ρ be a separable state as given by Eqn. (22). Then $D(\rho) \leq 1 - \max_i p_i \leq 1 - \frac{1}{n}$.*

Proof. Assume WLOG that $\max_i p_i = p_1$. Then $1/n \leq p_1 \leq 1$. Choose any $U_A \in \text{RU}(\mathcal{A})$ such that $|a_1\rangle$ is an eigenvector of U_A . Then any term in the double sum of Eqn. (23) in which $|a_1\rangle$ appears vanishes. We can hence loosely upper bound the value of Eqn. (23) by $\sqrt{(\sum_{i \neq 1, j \neq 1} p_i p_j)} = 1 - p_1$. Recalling that $p_1 \geq 1/n$ yields the desired bound. \square

When $n = 2$, i.e. when ρ is rank at most two, observe from Lem. 2 that $D(\rho) \leq 1/2$, and this is attainable only when $p_1 = p_2 = 1/2$. We now show that this bound can indeed be saturated, and characterize all states with $n = 2$ that do so.

Lemma 3. *Let ρ be a separable state as in Eqn. (22) with $p_1 = p_2 = 1/2$. Then $D(\rho) = 1/2$ if and only if $|\langle a_1 | a_2 \rangle| = 1/\sqrt{2}$ and $\langle b_1 | b_2 \rangle = 0$.*

Proof. Since by Lem. 1, $D(\rho)$ is invariant under local unitaries, we can assume without loss of generality that $|a_1\rangle = |0\rangle$, $|b_1\rangle = |0\rangle$, $|a_2\rangle = \cos \frac{\beta}{2} |0\rangle + \sin \frac{\beta}{2} |1\rangle$ and $|b_2\rangle = \cos \frac{\alpha}{2} |0\rangle + \sin \frac{\alpha}{2} |1\rangle$ for $\alpha, \beta \in [0, \pi]$, i.e. we can rotate the local states so as to eliminate relative phases. Further, since $U_A \in \text{RU}(\mathcal{A})$ in Eqn. (23), we can write $U_A = 2|u\rangle\langle u| - I$ for some $|u\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$, where $\theta, \phi \in [0, 2\pi]$. Via the latter, we can rewrite Eqn. (23) as:

$$\frac{1}{2} \sqrt{\sum_{i,j=1}^2 \langle b_i | b_j \rangle^2 (\langle a_i | a_j \rangle^2 - |\langle a_i | a_j \rangle - 2\langle a_i | u \rangle \langle u | a_j \rangle|^2)}. \quad (24)$$

Letting Δ denote the expression under the square root above, we have by substituting in our expressions for $|a_1\rangle$, $|a_2\rangle$, $|b_1\rangle$, $|b_2\rangle$, and $|u\rangle$ and algebraic manipulation that

$$\Delta = \cos^2 \frac{\alpha}{2} [2 \cos \beta \sin^2 \theta - \sin \beta \sin(2\theta) \cos \phi] + 1 + \sin^2 \theta - (\cos \beta \cos \theta + \sin \beta \sin \theta \cos \phi)^2. \quad (25)$$

Our goal is to maximize Δ with respect to α and β (which define ρ), and then minimize with respect to θ and ϕ (which define U_A). Observe now that choosing $\phi = \theta = 0$ reduces Eqn. (25) to $\Delta = 1 - \cos^2 \beta$. Hence, unless $\beta = \pi/2$ (i.e. $|\langle a_1 | a_2 \rangle| = 1/\sqrt{2}$), we can always achieve $D(\rho) < 1/2$. Thus, set $\beta = \pi/2$. Consider next $\phi = 0$, and leave θ unsigned. Then, Eqn. (25) reduces to $\Delta = 1 - \cos^2 \frac{\alpha}{2} \sin(2\theta)$, from which it is clear that unless $\alpha = \pi$ (i.e. $\langle b_1 | b_2 \rangle = 0$), we can always achieve $D(\rho) < 1/2$. Plugging these values of α and β into Eqn. (25), we have $\Delta = 1 + \sin^2 \theta \sin^2 \phi$, from which the claim follows. \square

Acknowledgements

We thank Gerardo Adesso, Dagmar Bruß, and Marco Piani for helpful discussions. Support from Canada's NSERC, CIFAR and MITACS programs is graciously acknowledged.

Note: After completion of this paper, the author learned of independent work in preparation on a similar topic [43].

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